CS103
Fall 2025

Lecture 19:

Context-Free Grammars

A Motivating Question

python3

```
>>> (137 + 42) - 2 * 3
173

>>> (60 + 37) + 5 * 8
137

>>> (200 / 2) + 6 / 2
103.0

>>>
```

Mad Libs for Arithmetic

(______ ______ ______) ______ ______ ______ ______

Int **Op** **Int** **Op** **Int** **Op** **Int**

This only lets us make arithmetic expressions of the form (**Int Op Int**) **Op Int Op Int**.

What about arithmetic expressions that don't follow this pattern?

Slide credit: Amy Liu

Recursive Mad Libs

`( int / ( int + int ) )`

Expr Op Expr Op Expr

What can an arithmetic expression be?

`int`	A single number.
Expr Op Expr	Two expressions joined by an operator.
(Expr)	A parenthesized expression.

A ***context-free grammar*** (or ***CFG***) is a recursive set of rules that define a language.

(There's a bunch of specific requirements about what those rules can be; more on that in a bit.)

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`
Expr → **Expr Op Expr**
Expr → `(`**Expr**`)`
Op → `+`
Op → `-`
Op → `×`
Op → `/`

This is called a *production rule*. It says "if you see **Expr**, you can replace it with **Expr Op Expr**."

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

This one says "if you see **Op**, you can replace it with `-`."

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Expr

⇒ **Expr Op Expr**

⇒ **Expr Op** `int`

⇒ `int` **Op** `int`

⇒ `int / int`

These red symbols are called ***nonterminals***. They're placeholders that get expanded later on.

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Expr
⇒ **Expr Op Expr**
⇒ **Expr Op** `int`
⇒ `int` **Op** `int`
⇒ `int / int`

The symbols in blue monospace are **terminals**. They're the final characters used in the string and never get replaced.

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Expr

⇒ **Expr Op Expr**

⇒ **Expr Op** `(`**Expr**`)`

⇒ **Expr Op** `(`**Expr Op Expr**`)`

⇒ **Expr** `×` `(`**Expr Op Expr**`)`

⇒ `int` `×` `(`**Expr Op Expr**`)`

⇒ `int` `×` `(int` **Op Expr**`)`

⇒ `int` `×` `(int` **Op** `int)`

⇒ `int × (int + int)`

Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - a set of ***nonterminal symbols*** (also called ***variables***),
  - a set of ***terminal symbols*** (the ***alphabet*** of the CFG),
  - a set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - a ***start symbol*** (which must be a nonterminal) that begins the derivation. By convention, the start symbol is the one on the left-hand side of the first production.

Expr → `int`
Expr → **Expr Op Expr**
Expr → `(`**Expr**`)`
Op → `+`
Op → `-`
Op → `×`
Op → `/`

Some CFG Notation

- In today's slides, capital letters in **Bold Red Uppercase** will represent nonterminals.
  - e.g. **A**, **B**, **C**, **D**
- Lowercase letters in `blue monospace` will represent terminals.
  - e.g. `t`, `u`, `v`, `w`
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - e.g. α, γ, ω
- You don't need to use these conventions on your own; just make sure whatever you do is readable.

A Notational Shorthand

Expr → `int` | **Expr Op Expr** | `(`**Expr**`)`

Op → `+` | `-` | `×` | `/`

Derivations

Expr
⇒ **Expr Op Expr**
⇒ **Expr Op** `(`**Expr**`)`
⇒ **Expr Op** `(`**Expr Op Expr**`)`
⇒ **Expr** `×` `(`**Expr Op Expr**`)`
⇒ `int × (`**Expr Op Expr**`)`
⇒ `int × (int` **Op Expr**`)`
⇒ `int × (int` **Op** `int)`
⇒ `int × (int + int)`

- A sequence of zero or more steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string α derives string ω, we write $\alpha \Rightarrow^* \omega$.
- In the example on the left, we see that

  Expr $\Rightarrow^*$ `int × (int + int)`.

Expr → `int` | **Expr Op Expr** | `(`**Expr**`)`
Op → `+` | `-` | `×` | `/`

The Language of a Grammar

- If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega} \}$$

- That is, $\mathscr{L}(G)$ is the set of strings of terminals derivable from the start symbol.

If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \omega \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \omega \}$$

Consider the following CFG G over Σ = {**a**, **b**, **c**, **d**}:

Q → **Qa** | **dH**
H → **bHb** | **c**

Which of the following strings are in $\mathscr{L}(G)$?

dca
dc
cad
bcb
dHaa

Answer at [***https://cs103.stanford.edu/pollev***](https://cs103.stanford.edu/pollev)

Context-Free Languages

- A language L is called a ***context-free language*** (or CFL) if there is a CFG G such that $L = \mathscr{L}(G)$.
- Questions:
  - How are context-free and regular languages related?
  - How do we design context-free grammars for context-free languages?

CFGs and Regular Expressions

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- You can use the symbols * and ∪ if you'd like in a CFG, but they just stand for themselves.
- Consider this CFG G:

$$\mathbf{S} \to \texttt{a*b}$$

- Here, $\mathscr{L}(G) = \{\texttt{a*b}\}$ and has cardinality one. That is, $\mathscr{L}(G) \neq \{\ \texttt{a}^n\texttt{b} \mid n \in \mathbb{N}\ \}$.

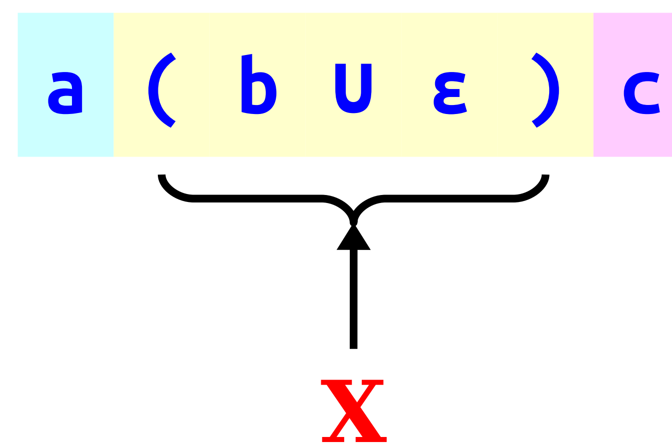
CFGs and Regular Expressions

- ***Theorem:*** Every regular language is context-free.
- ***Proof idea:*** Show how to convert an arbitrary regular expression into a context-free grammar.

S → **a**X**c**

X → **b** | ε

It's totally fine for a production to replace a nonterminal with the empty string.



CFGs and Regular Expressions

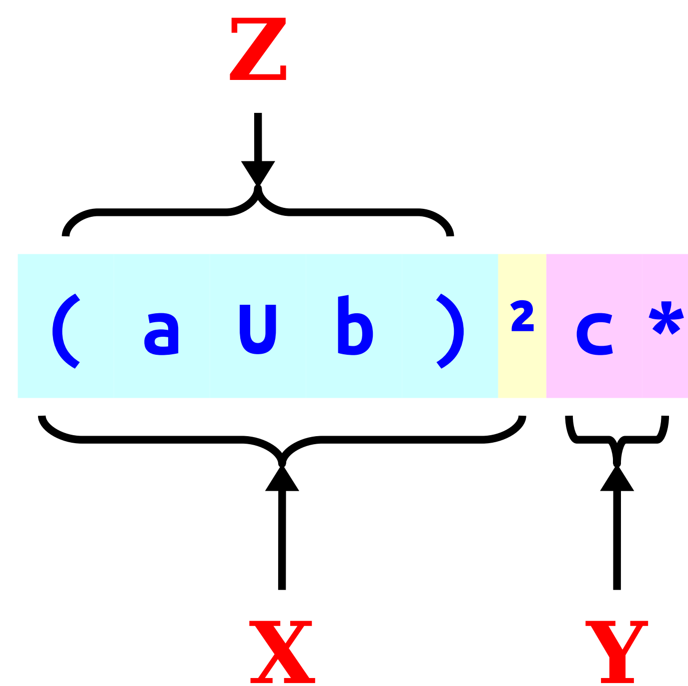
- ***Theorem:*** Every regular language is context-free.
- ***Proof idea:*** Show how to convert an arbitrary regular expression into a context-free grammar.

S → **XY**

X → **ZZ**

Z → a | b

Y → c**Y** | ε



The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

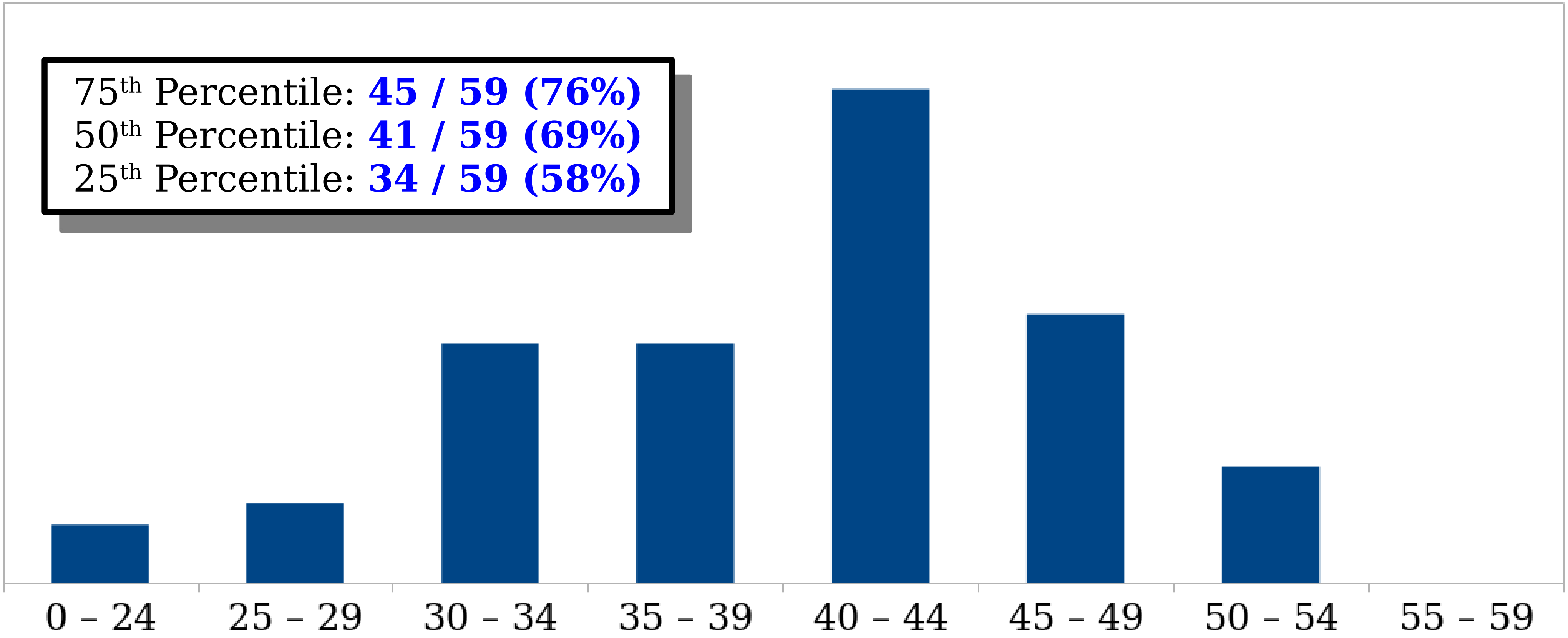
a	a	a	a	b	b	b	b

$$\mathscr{L}(G) = \{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$$


Regular Languages
CFLs
All Languages

Time-Out for Announcements!

Problem Set Five Graded



Problem Set Seven

- Problem Set Six was due today at 1:00PM.
  - You can extend the deadline to Saturday at 1:00PM using a late day.
- Problem Set Seven goes out today. It’s due next Friday at 1:00PM.
  - It’s all about regular expressions, properties of regular languages, and gives a first glimpse at nonregular languages.
  - We’ve tuned the length given that you have a midterm next Monday.
- As always, come talk to us if you have any questions.

Second Midterm Logistics

- Our second midterm exam is next ***Monday, November 10*** from ***7:00PM – 10:00PM***

Seating assignments have changed.

☞ ***Check the seating assignment page again.*** ☜

Write down your new seat.

- Topic coverage is primarily lectures 06 – 13 (functions through induction) and PS3 – PS5. Finite automata and onward won't be tested here.
    - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- The exam is closed-book and closed-computer. You can bring one double-sided 8.5" × 11" sheet of notes with you.
- Contact us ASAP if you need an alternate exam and haven't heard from us with date/time/place.

Our Advice

- ***Stay fed and rested.*** You are not a brain in a jar. You are a rich, complex, beautiful human being. Please take care of yourself.
- ***Read all questions before diving into them.*** You don't have to go sequentially. Read over each problem so you know what to expect, then pick whichever one looks easiest and start there.
- ***Reflect on how far you've come.*** How many of these questions would you have been able to *understand* two months ago? That's the mark that you're learning something!

Three Questions

- What's something you know now that, at the start of the quarter, you knew you didn't know?
- What's something you know now that, at the start of the quarter, you *didn't* know you didn't know?
- What's something you *don't* know now that, at the start of the quarter, you *didn't* know you didn't know?

Back to CS103!

Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.
- Check our online “Guide to CFGs” for more information about CFG design.
- We’ll hit the highlights in the rest of this lecture.

Designing CFGs

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for L by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If ω is a palindrome, then **a**ω**a** and **b**ω**b** are palindromes.
  - No other strings are palindromes.

$$S \rightarrow \varepsilon \mid \texttt{a} \mid \texttt{b} \mid \texttt{a}S\texttt{a} \mid \texttt{b}S\texttt{b}$$

Designing CFGs

- Let Σ = {**{**, **}**} and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced braces $\}$
- Some sample strings in L:

{{{}}}

{{}}{}

{{}{}}{{}{}}

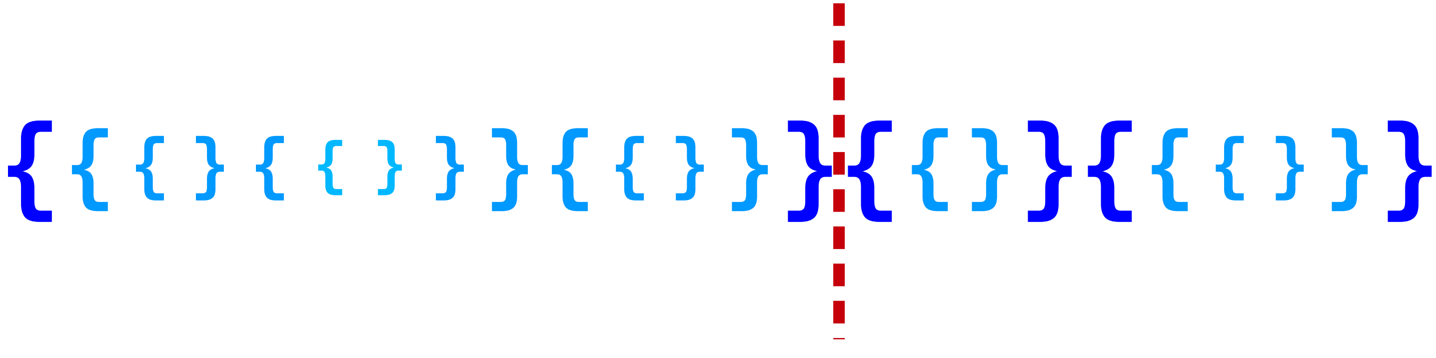
{{{{{}}}{{}}}}

ε

{}{}

Designing CFGs

- Let Σ = {**{**, **}**} and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced braces }
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace.



Designing CFGs

- Let $\Sigma = \{$**{**, **}**$\}$ and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced braces $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace. Removing the first brace and the matching brace forms two new strings of balanced braces.

S → {S}S | ε

Designing CFGs

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{ w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$

Which of these CFGs have language L?

S → aSb | bSa | ε

S → abS | baS | ε

S → abSba | baSab | ε

S → SbaS | SabS | ε

Answer at
https://cs103.stanford.edu/pollev

Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!
- You'll (most likely) design your own CFG for this language on Problem Set 8.

Designing CFGs

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{a, \stackrel{?}{=}\}$ and let $L = \{ a^n \stackrel{?}{=} a^n \mid n \in \mathbb{N} \}$.
- Is the following a CFG for L?

$$S \to X \stackrel{?}{=} X$$

$$X \to aX \mid \varepsilon$$

$$\begin{aligned} & S \\ \Rightarrow\ & X \stackrel{?}{=} X \\ \Rightarrow\ & aX \stackrel{?}{=} X \\ \Rightarrow\ & aaX \stackrel{?}{=} X \\ \Rightarrow\ & aa \stackrel{?}{=} X \\ \Rightarrow\ & aa \stackrel{?}{=} aX \\ \Rightarrow\ & aa \stackrel{?}{=} a \end{aligned}$$

Finding a Build Order

- Let $\Sigma = \{\mathtt{a}, \stackrel{?}{=}\}$ and let $L = \{\mathtt{a}^n \stackrel{?}{=} \mathtt{a}^n \mid n \in \mathbb{N}\}$.
- To build a CFG for L, we need to be more clever with how we construct the string.
  - If we build the strings of a's independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of a's at the same time.
- Here's one possible grammar based on that idea:

$$\mathbf{S} \to \stackrel{?}{=} \mid \mathtt{a}\mathbf{S}\mathtt{a}$$

$$
\begin{aligned}
& \mathbf{S} \\
\Rightarrow\ & \mathtt{a}\mathbf{S}\mathtt{a} \\
\Rightarrow\ & \mathtt{aa}\mathbf{S}\mathtt{aa} \\
\Rightarrow\ & \mathtt{aaa}\mathbf{S}\mathtt{aaa} \\
\Rightarrow\ & \mathtt{aaa}\stackrel{?}{=}\mathtt{aaa}
\end{aligned}
$$

Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

Applications of Context-Free Grammars

CFGs for Programming Languages

```
BLOCK  → STMT
       | { STMTS }

STMTS  → ε
       | STMT STMTS

STMT   → EXPR;
       | if (EXPR) BLOCK
       | while (EXPR) BLOCK
       | do BLOCK while (EXPR);
       | BLOCK
       | …

EXPR   → identifier
       | constant
       | EXPR + EXPR
       | EXPR – EXPR
       | EXPR * EXPR
       | …
```

Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program "means."
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? ***Take CS143!***

Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- The ***Stanford Parser*** project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

Next Time

- ***No Class Monday (Midterm 2)***
- ***Then, when we get back…***
  - ***Turing Machines***
    - What does a computer with unbounded memory look like?
    - How would you program it?